THE CARTAN-KARLHEDE ALGORITHM FOR GÖDEL-LIKE SPACETIMES

D. BROOKS[1], D.D. MCNUTT[1], J.P. SIMARD[2], AND N. MUSOKE[3]

[1]DEPARTMENT OF MATHEMATICS AND STATISTICS,
DALHOUSIE UNIVERSITY, HALIFAX, NOVA SCOTIA,
CANADA B3H 3J5
[2] DEPARTMENT OF MATHEMATICS AND STATISTICS,
UNIVERSITY OF NEW BRUNSWICK, FREDERICTON, NEW BRUNSWICK
CANADA, E3B 5A3
[3] PERIMETER INSTITUTE FOR THEORETICAL PHYSICS,
WATERLOO, ONTARIO
CANADA, N2L 2Y5

ABSTRACT. We present an improvement to the equivalence algorithm for three-dimensional gravity utilizing a three-dimensional analogue of the Newman-Penrose spinor formalism. To illustrate this algorithm we classify the entire class of Gödel-like spacetimes. After introducing the key quantities required for the calculation of the curvature for the Gödel-like spacetimes, we summarize the permitted Segre types for the Ricci tensor and list the constraints for the metric functions in each case. With this first step towards classification we continue the Cartan-Karlhede algorithm for each case. As an application, we express the polynomial scalar curvature invariants for Segre type [11,1] Gödel-like spacetimes in terms of the Cartan invariants.

1. INTRODUCTION

The equivalence problem in general relativity is the issue of determining whether two spacetimes can be transformed from one to another by some change of coordinates. This is an important issue for solutions to the Einstein field equations, as one would like to know when two solutions are related and hence describe the same gravitational field. It can also be difficult to determine whether an effect derived from the metric is due to the coordinates chosen or is of a real physical nature. The solution to the equivalence problem concerns itself with obtaining an invariant classification of the local geometry of the spacetime. Such a solution provides a framework from which the answers to these and other questions can be obtained [2]. In general, a solution to the problem can be quite applicable and have far-reaching consequences [3, 4, 5].

Mathematically, interest in the equivalence problem goes back to the time of Gauss, who more than likely knew the answer in dimension 2 [3]. Christoffel was

the first to investigate this problem for n-dimensional Riemannian manifolds admitting no symmetries, and his work implied that for $n = 4$ the twentieth covariant derivative of the curvature tensor was required to determine equivalence [6]. Elie Cartan applied his method of moving frames in 1946 to the classification of Lie pseudo-groups. Cartan also showed that this approach could be extended to metrics admitting symmetries, and in general required a comparison of the curvature tensor components and its covariant derivatives up to $n(n + 1)/2$ order [3, 7]. Sternberg completed Cartan's approach in the 1960's, and with the development of computer algebra systems Anders Karlhede gave Cartan's approach a true algorithmic form and adapted it specifically for general relativity in 1980 [8, 9]. Algorithms using the algebraic classification of the irreducible parts of the curvature spinor and the Newman-Penrose spinor formalism have also been developed, implemented for various computer algebra suites [10, 11, 12].

In this paper, we examine the geometry of the three-dimensional Gödel-like spacetimes using an adaptation of the Cartan-Karlhede (C.K.) algorithm to three-dimensional spacetimes, which begins by summarizing the permitted Segre types for the Ricci tensor of the Gödel-like spacetimes; for each Segre type we list the constraints for the metric functions. Using Milson and Wylleman's three dimensional spinor formalism [14], we continue the Cartan-Karlhede classification algorithm by studying the covariant derivatives of the curvature tensor for each Segre type.

For the remainder of this section, we introduce the key quantities required for the calculation of the local curvature for the Gödel-like spacetimes. In section 2, we present a collection of lemmas detailing the Segre types for the Ricci tensor and outline the explicit Cartan-Karlhede algorithm along with completing the 0th iteration of the algorithm. In section 3, we examine the 1st iteration of the algorithm, and provide the formula for the covariant derivative of the Ricci tensor. This will allow for five separate cases for the algorithm, with each detailing the equivalence algorithm for a possible Segre type for these spacetimes. In Section 4, we conclude the paper with some applications of the Cartan invariants, and their benefit in the study of scalar curvature invariants.

1.1. **The Coframe Formalism.** The three-dimensional Gödel metric is defined as [2]

$$ds^2 = -[dt + H(r)d\phi]^2 + D^2(r)d\phi^2 + dr^2.$$

Using the formalism in [14] and defining $F_\pm = H \pm D$, the coframe $\{\theta^a\}$ may be written in compact form:

$$
\begin{aligned}
-n_\mu = \theta^0{}_\mu &= -\frac{1}{\sqrt{2}}(dt + F_-d\phi), \\
m_\mu = \theta^1{}_\mu &= \sqrt{2}dr, \\
-\ell_\mu = \theta^2{}_\mu &= -\frac{1}{\sqrt{2}}(dt + F_+d\phi).
\end{aligned}
$$

The Cartan structure equations for a symmetric connection are

$$
\begin{aligned}
d\theta^a &= -\omega^a{}_b \wedge \theta^b, \\
d\omega^a{}_b &= -\frac{1}{2}\omega^a{}_c \wedge \omega^c{}_b + \Omega^a{}_b,
\end{aligned}
$$

where $\omega^a_{\ b}$, $\Omega^a_{\ b}$ are the connection one-form and curvature two-form:

$$\omega^a_{\ b} = \Gamma^a_{\ cb}\theta^c,$$
$$\Omega^a_{\ b} = \frac{1}{2}R^a_{\ bcd}\theta^c \wedge \theta^d.$$

We note that the connection one-form satisfies $\omega_{ab} = -\omega_{ba}$.

The connection one-forms arising from the first Cartan structure equation are then:

$$\omega_{01} = \frac{F'_+}{2\sqrt{2}D}\theta^0 - \frac{D'}{2\sqrt{2}D}\theta^2,$$
$$\omega_{02} = \frac{H'}{2\sqrt{2}D}\theta^1,$$
$$\omega_{12} = \frac{D'}{2\sqrt{2}D}\theta^0 + \frac{F'_-}{2\sqrt{2}D}\theta^2.$$

Applying the second Cartan structure equation to the connection one-forms, we compute the following curvature two-forms:

$$R^1_{\ 0} = -\frac{1}{2}\left[\frac{F''_+}{D} - \left(\frac{H'}{D}\right)^2 - \frac{D'H'}{D^2}\right]\theta^1 \wedge \theta^0 + \frac{1}{4}\left[\frac{2D''}{D} - \left(\frac{H'}{D}\right)^2\right]\theta^1 \wedge \theta^2,$$
$$R^1_{\ 2} = \frac{1}{4}\left[\frac{2D''}{D} - \left(\frac{H'}{D}\right)^2\right]\theta^1 \wedge \theta^0 + \frac{1}{2}\left[\frac{F''_-}{D} - \left(\frac{H'}{D}\right)^2 - \frac{D'H'}{D^2}\right]\theta^1 \wedge \theta^2,$$
$$R^2_{\ 2} = \frac{1}{4}\left[\left(\frac{H'}{D}\right)^2\right]\theta^0 \wedge \theta^2.$$

The non-zero components of the Ricci tensor are then:

$$Ric_{00} = -\frac{1}{4}\left[\frac{2D''}{D} - \left(\frac{H'}{D}\right)^2\right],$$
$$Ric_{11} = -\frac{1}{2}\left[\frac{F''_+}{D} - \left(\frac{H'}{D}\right)^2 - \frac{D'H'}{D^2}\right],$$
$$Ric_{12} = \frac{1}{2}\left[\frac{D''}{D}\right],$$
$$Ric_{22} = \frac{1}{2}\left[\frac{F''_-}{D} + \left(\frac{H'}{D}\right)^2 - \frac{D'H'}{D^2}\right],$$

where we have lowered the upper-index in the Riemann tensor to exploit the anti-symmetry of the first pair of the indices. There is one more component that is relevant to the construction of the Riemann tensor, and it comes from contracting the Ricci tensor, $R = Ric^\alpha_{\ \alpha}$ - the Ricci scalar:

$$R = -\frac{1}{2}\left[\frac{4D''}{D} - \left(\frac{H'}{D}\right)^2\right].$$

To bring this in line with the quantities defined in [14] for the trace-free part of the Ricci tensor $S_{ab} = Ric_{ab} - \frac{R}{3}g_{ab}$, we note:

$$
\begin{aligned}
\Psi_0 &= S_{\mu\nu}\ell^\mu\ell^\nu = S_{00}, \\
\Psi_1 &= S_{\mu\nu}m^\mu\ell^\nu = S_{10}, \\
\Psi_3 &= S_{\mu\nu}m^\mu n^\nu = S_{12}, \\
\Psi_2 &= S_{\mu\nu}m^\mu m^\nu = S_{ab}n^a\ell^b = S_{11} = S_{20}, \\
\Psi_4 &= S_{\mu\nu}n^\mu n^\nu = S_{22}.
\end{aligned}
$$

For the Gödel-like spacetimes we find that $\Psi_1 = \Psi_3 = 0$ and the non-zero components are:

(1)
$$
\begin{aligned}
\Psi_0 &= -\frac{1}{2}\left[\frac{D''}{D} - \left(\frac{H'}{D}\right)^2 - \left(\frac{H'}{D}\right)'\right] \\[2mm]
\Psi_2 &= -\frac{1}{6}\left[\frac{D''}{D} - \left(\frac{H'}{D}\right)^2\right] \\[2mm]
\Psi_4 &= -\frac{1}{2}\left[\frac{D''}{D} - \left(\frac{H'}{D}\right)^2 + \left(\frac{H'}{D}\right)'\right] \\[2mm]
R &= -\frac{1}{2}\left[\frac{4D''}{D} - \left(\frac{H'}{D}\right)^2\right].
\end{aligned}
$$

To normalize the components of the Ricci tensor we must use the Lorentz freedom to bring the components in line with the table in [2] on page 10. To summarize the frame freedoms we have a boost:

$$
\tilde{\theta}^0 = \sqrt{A}\theta^0, \ \tilde{\theta}^1 = \theta^1, \ \tilde{\theta}^2 = \frac{1}{\sqrt{A}}\theta^2,
$$

and null rotations about n^a and ℓ^a respectively:

$$
\tilde{\theta}^0 = \theta^0, \ \tilde{\theta}^1 = \theta^1 + B\theta^0, \ \tilde{\theta}^2 = \theta^2 + B\theta^1 + \frac{B^2}{2}\theta^0,
$$

$$
\tilde{\theta}^0 = \theta^0 + C\theta^1 + \frac{C^2}{2}\theta^2, \ \tilde{\theta}^1 = \theta^1 + C\theta^2, \ \tilde{\theta}^2 = \theta^2,
$$

While we will work primarily with these Lorentz transformations, at times it will be helpful to express the null coframe as an orthonormal coframe $\{t, x, m\}$ and rotate the spatial one-forms x and m.

2. Segre types for the Gödel-like Spacetimes: The 0-th iteration of the C.K. Algorithm

Noting that for the Gödel spacetimes $\Psi_1 = \Psi_3 = 0$, we have the following distinct subclasses

Lemma 2.1. *In general, the Gödel-like spacetimes will have a Ricci tensor of Segre type [11,1] (or P- type I) if there is no coordinate system where $D(r)$ and $H(r)$ are any pair of functions not satisfying the equations in the ensuing lemmas. Applying a boost, the components of the Ricci spinor are:*

$$
\tilde{\Psi}_0 = \tilde{\Psi}_4 = \sqrt{|\Psi_0\Psi_4|}, \ \tilde{\Psi}_2 = \Psi_2, \ \tilde{R} = R
$$

where the original components are defined in equation (1).

Lemma 2.2. *In general, the Gödel-like spacetimes with a Ricci tensor of Segre type [1 $z\bar{z}$] (or P- type IZ) admit a coordinate system in which the metric functions $D(r)$ and $H(r)$ satisfy the following constraint:*

$$\frac{H'}{D} = f(r), \ \frac{D''}{D} - f^2 = 0, \ f' \neq 0.$$

The components of the Ricci spinor are:

$$\Psi_0 = -\Psi_4 = \frac{1}{2}f', \ \Psi_2 = 0, \ R = -\frac{3}{2}f,$$

where the original components are defined in equation (1).

Lemma 2.3. *Those Gödel-like spacetimes with a Ricci tensor with Segre type [12](or P-type IIZ) admit a coordinate system in which the metric functions $D(r)$ and $H(r)$ satisfy the following constraints:*

$$\frac{H'}{D} = f(r), \ \frac{D''}{D} - f^2 + f' = 0 \ and \ f' \neq 0.$$

Applying a boost, the components of the Ricci Spinor are then,

$$\tilde{\Psi}_0 = 1, \ \tilde{\Psi}_4 = 0, \ \tilde{\Psi}_2 = \frac{f'}{6}, \ \tilde{R} = -\frac{1}{2}[3f^2 + 4f'].$$

Lemma 2.4. *Those Gödel-like spacetimes with a Ricci tensor with Segre type [(11),1] (or P-type DZ) admit a coordinate system in which the metric functions $D(r)$ and $H(r)$ satisfy the following constraint:*

$$H' = CD, \ C \in \mathbb{R}.$$

The components of the Ricci Spinor are then,

$$\Psi_0 = \Psi_4 = 3\Psi_2 = -\frac{1}{2}\left[\frac{D''}{D} - C^2\right], \ R = -\frac{1}{2}\left[\frac{4D''}{D} - C^2\right].$$

Lemma 2.5. *Those Gödel-like spacetimes with a Ricci tensor with Segre type [(11,1)] (or P-type O) admit a coordinate system in which the metric functions $D(r)$ and $H(r)$ satisfy the following constraints:*

$$H' = CD, \ D'' - C^2 D = 0, \ C \in \mathbb{R}$$

The components of the Ricci Spinor are then,

$$\Psi_0 = \Psi_4 = 3\Psi_2 = 0, \ R = -\frac{3C^2}{2}.$$

To put the Segre types into context, we reiterate the five steps of the Karlhede algorithm for the zeroth iteration, $q = 0$:

(1) Calculate the set I_0, i.e., the derivatives of the curvature up to the 0-th order, i.e. the Ricci tensor.
(2) Fix the frame as much as possible by putting the elements of I_0 into canonical form.
(3) Find the frame freedom given by the isotropy group H_0 of transformations which leave invariant the normal form. [1] of I_0.
(4) Find the number t_0 of functionally independent functions of spacetime coordinates in the elements of I_0, brought into the normal form.

---

[1]What we are calling the normal form of the Ricci tensor, Sousa et al calls the canonical form.

(5) Without replacing q with 0, the last step says "If the isotropy group H_q is the same as H_{q-1} and the number of functionally independent functions t_0 is equal to t_{q-1} , then let $q = p + 1$ and stop". As there are no previous steps, we must always set $q = 1$ and start on the 1st iteration of the C.K. algorithm.

By studying the effect of the members of the Lorentz group $SO(1,2)$ which **do** change the form of the metric, we may use these frame transformations to fix the normal frame and determine all possible Segre types.

As an example, in the general case of the Gödel spacetimes with Ricci tensor of Segre type [11,1], the null rotations about ℓ and n and boosts are fixed with $B = C = 0$ and $A = \sqrt{\frac{\Psi_0}{\Psi_4}}$ (or $A = \sqrt{\frac{-\Psi_0}{\Psi_4}}$ in the case of P-type IZZ)[2] . To produce the remaining Segre types we examine when the choices for the parameters A, B and C are no longer applicable. For example, if Ψ_0 or Ψ_4 vanish, we can no longer use the boost we had intended, as it would require a division by zero; thus we have produced the class of Gödel spacetimes admitting a Ricci tensor with Segre type [12]. The Ricci tensor is of Segre Type [(11),1] arises when $A = 1$ (the identity element for boosts) and the null rotations about ℓ and n produce a frame transformation leaving the Ricci tensor invariant for some choice of parameters $B(\theta)$ and $C(\theta)$ (i.e., an element of SO(2) whose effect is a rotation of the spatial vectors). Finally those Gödel-like spacetimes with a Ricci tensor of Segre type [(11,1)] are those in which all of $SO(1,2)$ leaves the Ricci tensor unchanged.

Thus, by determining all of the possibilities for the Segre type of the Ricci tensor and recording the conditions on the metric functions $H(r)$ and $D(r)$ we have begun five separate instances of the C.K. algorithm. In each case we use the components of the Ricci tensor in normal form to produce simpler invariants, and as they are all functions of r we have *at most one functionally independent function*; thus, determining t_0 in each instance is easily done. The dimension of the isotropy group, H_0, of I_0 (the components of the Ricci tensor) is found by counting the remaining frame freedom. Step 1-4 have been completed for the 0-th iteration. To continue, we set $q = 1$ and evaluate the covariant derivative of the Ricci tensor to begin computing I_1.

3. The 1st iteration of the C.K. algorithm

The next iteration of the C.K. algorithm requires that we must compute the covariant derivative of the Ricci tensor, or equivalently the Ricci spinor, although we will focus on tensors due to the simplicity of the metric. It is here, where the spin coefficients are introduced as potential invariants. This can be seen by using the formula for the covariant derivative [3]:

$$(2) \qquad R_{ab;c} = \nabla_c R_{ab} = R_{ab,c} - R_{db}\Gamma^d_{ca} - R_{ad}\Gamma^d_{cb},$$

or equivalently the Cotton-York tensor,

$$(3) \qquad C_a{}^b = -\sqrt{-|g|}\epsilon^{bcd}(R_{ca} - \frac{R}{4}g_{ca})_{,d}.$$

However, there is a problem: the connection coefficients used to compute the Ricci tensor prior to any frame transformations may be different from the coframe used

---

[2]i.e. both are fixed as the identity element of SO(1,2)
[3]Henceforth we will write $Ric_{ab} = R_{ab}$.

to produce the normal form of the Ricci tensor. For example, in those Gödel-like spacetimes with Ricci tensor of Segre type [11,1] or [12] one must boost the coframe to put the Ricci tensor in normal form. On the other hand, those Gödel-like spacetimes with Segre type [(11),1] and [(11,1)] require no frame transformations whatsoever.

To illustrate the usefulness of this calculation, we will repeat the analysis of the C.K. algorithm for those Gödel-like spacetimes admitting a Ricci tensor with each P-type listed above.

3.1. **P-type O.** As all components of the Ricci tensor are constant, using Lemma (2.5) one may show that the covariant derivative of the Ricci tensor is zero. We conclude that these are locally homogeneous spaces.

3.2. **P-type I.** According to Lemma (2.1) at zeroth order the set of functions $\{R, \Psi_2, \Psi_0\}$ are all non-constant invariants. With some algebra, these give rise to a simpler set of invariants $\left\{ (\frac{H'}{D})^2, \frac{D''}{D}, \left(\frac{H'}{D}\right)' \right\}$; the number of functionally independent invariants is $t_0 = 1$ as these are all functions of r alone. Thus, we have fixed all of the isotropy at first order ($dim H_0 = 0$) and produced one functionally independent invariant and two *algebraically independent* essential classifying functions,

$$\{I_0, F_0(I_0), F_1(I_0)\} = \left\{ \left(\frac{H'}{D}\right)^2, \frac{D''}{D}, \left(\frac{H'}{D}\right)' \right\}.$$

We proceed to the next iteration of the algorithm to show that the algorithm must terminate, since $dim H_1 = 0$ and $t_1 = 1$, and more importantly, to collect all essential classifying functions. As all isotropy has been fixed, the metric coframe is an invariant coframe, and so the frame derivatives of all 0-th order invariants are now invariants as well. Using this fact we may solve for the spin-coefficients [14] from the components of the covariant derivative of the Ricci tensor:

$$\kappa = \frac{F'_+}{2\sqrt{2}D}, \ \sigma = 0, \ \tau = \frac{D'}{2\sqrt{2}D},$$

$$\epsilon = 0, \ 2\alpha = \frac{H'}{2\sqrt{2}D}, \ \gamma = 0,$$

$$\pi = \frac{D'}{2\sqrt{2}D}, \ \lambda = 0, \ \nu = \frac{F'_-}{2\sqrt{2}D}.$$

Since the identities $\tau = -\pi$, $\kappa = 2\alpha + \tau$, and $\nu = 2\alpha - \tau$ hold, we see that only two algebraically independent invariants appear in the spin-coefficients; however, as $I_0 - 32\alpha^2 = 0$, we need only include $F_3 = 2\sqrt{2}\tau$ as the new algebraically independent invariant. Returning to the frame derivatives, we have two algebraically independent invariants appearing, namely δF_0 and δF_1. Thus the list of algebraically independent invariants required to classify the spacetime is:

$$\{I_0, F_0, F_1; F_2, F_3, F_4\} = \left\{ \left(\frac{H'}{D}\right)^2, \frac{D''}{D}, \left(\frac{H'}{D}\right)'; \frac{D'}{D}, \left(\frac{D''}{D}\right)', \left(\frac{H'}{D}\right)'' \right\}.$$

3.3. **P-type IZ.** According to Lemma (2.2), at zeroth order the set of functions $\{R, \Psi_0\}$ are all non-constant invariants. These give rise to a simpler set of invariants:

$$\{I_0, F_0(I_0)\} = \left\{ \left(\frac{H'}{D}\right)^2, \ \left(\frac{H'}{D}\right)' \right\}.$$

The number of functionally independent invariants is $t_0 = 1$ as these are all functions of r alone. Thus, we have fixed all of the isotropy at first order ($dimH_0 = 0$) and produced one functionally independent invariant and one algebraically independent essential classifying function.

We proceed to the next iteration of the algorithm to show that the algorithm must terminate, as $dimH_1 = 0$ and $t_1 = 1$, and more importantly, to collect all essential classifying functions. As all isotropy has been fixed, the metric coframe is an invariant coframe, and we may solve for the spin-coefficients [14] from the components of the covariant derivative of the Ricci tensor:

$$\kappa = \frac{F'_+}{2\sqrt{2}D}, \ \sigma = 0, \ \tau = \frac{D'}{2\sqrt{2}D},$$

$$\epsilon = 0, \ 2\alpha = \frac{H'}{2\sqrt{2}D}, \ \gamma = 0,$$

$$\pi = \frac{D'}{2\sqrt{2}D}, \ \lambda = 0, \ \nu = \frac{F'_-}{2\sqrt{2}D}.$$

Yet again, due to the identities $\tau = -\pi$, $\kappa = 2\alpha + \tau$, $\nu = 2\alpha - \tau$, and $I_0 - 4\sqrt{2}\alpha = 0$, we need only include $F_3 = 2\sqrt{2}\tau$ as the new algebraically independent invariant. Returning to the frame derivatives, we have one algebraically independent invariants appearing, δF_0. Thus, the list of algebraically independent invariants required to classify the spacetime is:

$$\{I_0, F_0; F_1 F_2\} = \left\{ \left(\frac{H'}{D}\right)^2, \left(\frac{H'}{D}\right)'; \frac{D'}{D}, \left(\frac{H'}{D}\right)'' \right\}.$$

3.4. **P-type IIZ.** From Lemma (2.3), at zeroth order the set of functions $\{R, \Psi_2\}$ are all non-constant invariants. These give rise to a simpler set of invariants $\{f^2, \ f'\}$. The number of functionally independent invariants is $t_0 = 1$. Thus, we have fixed all of the isotropy at zeroth order, and we have produced one functionally independent invariant and an algebraically independent classifying function:

$$\{I_0, F_0(I_0)\} = \left\{ f^2, \ f' \right\}.$$

At the next iteration of the algorithm, the metric coframe is an invariant coframe, and so we may solve for the spin-coefficients [14] from the components of the covariant derivative of the Ricci tensor:

$$\kappa = \frac{F'_+}{2\sqrt{2}D}, \ \sigma = 0, \ \tau = \frac{D'}{2\sqrt{2}D},$$

$$\epsilon = 0, \ 2\alpha = \frac{H'}{2\sqrt{2}D}, \ \gamma = 0,$$

$$\pi = \frac{D'}{2\sqrt{2}D}, \ \lambda = 0, \ \nu = \frac{F'_-}{2\sqrt{2}D}.$$

We need only include $F_3 = 2\sqrt{2}\tau$ as the new algebraically independent invariant as the same identities $\tau = -\pi$, $\kappa = 2\alpha + \tau$, $\nu = 2\alpha - \tau$, and $I_0 - 4\sqrt{2}\alpha = 0$

hold as before. We have one algebraically independent invariant arising from the frame derivatives of the invariants, δF_0. Thus, the list of algebraically independent invariants required to classify the spacetime is:

$$\{I_0, F_0; F_1 F_2\} = \left\{ f^2, f'; \frac{D'}{D}, f'' \right\}.$$

.

3.5. **P-type DZ.** Lemma (2.4) indicates that at zeroth order the set of functions $\{R, \Psi_2\}$ are all non-constant invariants, which give rise to a simpler set of invariants $\left\{ \left(\frac{H'}{D}\right)^2 = C^2, \frac{D''}{D} \right\}$. The number of functionally independent invariants is $t_0 = 1$. Thus, we have fixed most of the isotropy at first order except spatial rotations, i.e., $dim H_0 = 1$, and we have produced one functionally independent invariant and a constant classifying function:

$$\{I_0, C^2\} = \left\{ \frac{D''}{D}, \left(\frac{H'}{D}\right)^2 \right\}.$$

At the first iteration the metric coframe is not yet an invariant coframe, and so one must be careful with taking frame derivatives of invariants, until an invariant coframe is discovered. By simplifying the components of covariant derivative of the Ricci tensor, one finds the first order invariants are $\delta\Psi_2, \delta R$, and the non-zero spin coefficients.

To entirely fix the frame, we consider the transformation rules for the spin-coefficients under an element of $SO(2)$:

$$(\ell + n)' = LHS, \ (\ell - n \pm 2im)' = e^{\mp it}(\ell - n \pm 2im).$$

This produces the following transformation rules for the spin coefficients and curvature components [14]:

$$(\gamma + \sigma - \epsilon - \lambda)' = LHS,$$
$$(4\alpha + \kappa - \pi + \nu - \tau)' = LHS,$$
$$(2(\gamma + \epsilon) \pm i(\kappa - \pi + \tau - \nu))' = e^{\pm it} LHS,$$
$$(4\alpha + \pi - \kappa + \tau - \nu + \pm 2o(\epsilon - \gamma + \sigma - \lambda))' = e^{\pm 2it} LHS,$$
$$(\lambda + \sigma - \gamma - \epsilon + \pm i(\pi - \tau))' = e^{\pm it}(LHS - \delta t \mp \tfrac{i}{2}(Dt - \Delta t)),$$
$$(\kappa + \pi + \nu + \tau)' = LHS - (Dt + \Delta t),$$
$$(\Psi_0 + 2\Psi_2 + \Psi_4)' = LHS,$$
$$(\Psi_0 - \Psi_4 \pm 2i(\Psi_1 + \Psi_3))' = e^{\mp it} LHS,$$
$$(|Psi_0 - 6\Psi_2 + \Psi_4 \pm 4i(\Psi_1 - \Psi_3))' = e^{\mp 2it} LHS.$$

Substituting the non-zero spin-coefficients and curvature components

$$\kappa = \frac{F'_+}{2\sqrt{2}D}, \ \sigma = 0, \ \tau = \frac{D'}{2\sqrt{2}D},$$
$$\epsilon = 0, \ 2\alpha = \frac{H'}{2\sqrt{2}D}, \ \gamma = 0,$$
$$\pi = \frac{D'}{2\sqrt{2}D}, \ \lambda = 0, \ \nu = \frac{F'_-}{2\sqrt{2}D},$$

we find the simpler transformation rules for the Gödel-like spacetimes:

$$2\gamma' = 0, \ \sigma' - \lambda' = 0, \ 8\alpha' = 8\alpha, \ \nu' = 2\alpha' + \tau', \ \kappa' = 2\alpha - \tau',$$
$$2[\pi' + \tau'] = -Dt - \Delta t, \ 2\sigma' \pm i(\pi' - \tau') = e^{\pm it}[-\delta t \mp i(2\tau - \frac{Dt}{2} + \frac{\Delta t}{2})]$$

Comparing with the spin-coefficients of original frame, $\tau = -\pi$, $\kappa = 2\alpha + \tau$, and $\nu = 2\alpha - \tau$, we conclude that setting the rotation parameter t to zero is the best choice: since any choice of $t \neq 0$ would cause $\sigma \neq 0$, $\tau \neq \pi$, and cause the frame derivatives D and Δ to be non-zero, effectively increasing the number of invariants instead of decreasing the number.

Fixing $t = 0$, we see that only two algebraically independent invariants appear in the spin-coefficients. However, as $C - 4\sqrt{2}\alpha = 0$, we need only include $F_3 = 2\sqrt{2}\tau$ as the new algebraically independent invariant - although, as C is a constant, α provides a useful discrete invariant, the sign of the constant C. Returning to the frame derivatives, we have one algebraically independent invariant appearing: δF_1. Thus the list of algebraically independent invariants required to classify the spacetime is:

$$\{C^2, I_0; F_0; sgn(C), F_1, F_2\} = \left\{ \left(\frac{H'}{D}\right)^2, \frac{D''}{D}; sgn\left(\frac{H'}{D}\right), \frac{D'}{D}, \left(\frac{D''}{D}\right)' \right\}.$$

3.5.1. *The 2nd iteration of the C.K. algorithm: P-type DZ.* Applying the frame derivatives to the first order invariants, we only produce one new algebraically independent classifying function:

$$\{C^2, I_0; F_0; sgn(C), F_1, F_2, F_3\} = \left\{ \left(\frac{H'}{D}\right)^2, \frac{D''}{D}; sgn\left(\frac{H'}{D}\right), \frac{D'}{D}, \left(\frac{D''}{D}\right)'; \left(\frac{D''}{D}\right)'' \right\}$$

4. APPLICATION TO SCALAR CURVATURE INVARIANTS

There are two major approaches for generating invariants: the Cartan-Karlhede algorithm and scalar polynomial invariants. We have seen that implementing the Cartan-Karlhede algorithm is not a trivial procedure. In comparison the computation of scalar polynomial invariants is straightforward, one merely takes the copies of the curvature tensor and its covariant derivatives, and contracts them to produce scalars. This can be done in any frame basis.

While this procedure is simple, there are several unanswered questions relating to the minimal number of scalar curvature invariants to compute, and the highest order covariant derivative of the curvature tensor needed to completely classify a spacetime. In four dimensions, considerable effort has been focused on studying the zeroth order scalar curvature invariants in order to determine the minimal basis of algebraically independent invariants needed to generate any other scalar curvature

invariant [16]. While less has been done to study higher order scalar curvature invariants [1, 15].

We will compare the Cartan invariants of the P-type I 3D Gödel-like metric to some of its scalar curvature invariants as given in [15]. In section 3.2 we found that for the P-type I metrics the algebraically independent invariants required for classification are

$$(4) \quad \{I_0, F_0, F_1; F_2, F_3, F_4\} = \left\{ \left(\frac{H'}{D}\right)^2, \frac{D''}{D}, \left(\frac{H'}{D}\right)'; \frac{D'}{D}, \left(\frac{D''}{D}\right)', \left(\frac{H'}{D}\right)'' \right\}.$$

Since the Ricci tensor is constructed from the metric, inverse metric, and derivatives of the metric, we expect the scalar curvature invariants also to be some rational functions of the the metric functions $H(r)$ and $D(r)$ and their derivatives. Calculating them, we find this to be the case. In particular, after making the substitutions

$$(5) \qquad \frac{H'}{D} = \sqrt{I_0}$$

$$(6) \qquad \frac{D''}{D} = F_0$$

$$(7) \qquad \frac{H''}{D} = \frac{H''}{D} - \left(\frac{H'}{D}\right)' + \left(\frac{H'}{D}\right)'$$
$$= \frac{H''}{D} - \left(\frac{H''}{D} - \frac{H'}{D}\frac{D'}{D}\right) + \left(\frac{H'}{D}\right)' = \sqrt{I_0}F_2 + F_1$$

for the zeroth order Cartan invariants, the first three of the zeroth order polynomial invariants become

$$(8) \qquad R^a{}_a = -1/2\frac{4\,(D)\,D'' - H'^2}{D^2} = -2\,F_0 + 1/2\,I_0$$

$$(9) \qquad R^{ab}R_{ab} = 2\,F_0{}^2 - 2\,F_0 I_0 + 3/4\,I_0{}^2 - 1/2\,F_1{}^2$$

$$(10) \quad R^{ab}R_a{}^c R_{bc} = -2\,F_0{}^3 + 3\,F_0{}^2 I_0 - 3/2\,F_0 I_0{}^2 + 3/4\,F_0 F_1{}^2 + 1/8\,I_0{}^3;$$

polynomials in $\sqrt{I_0}, F_0, F_1$. Note that each of the zeroth order Cartan invariants appears in the zeroth order polynomial invariants listed, and that no other functions do.

At the next order, we make the similar substitutions

$$(11) \qquad \frac{D'}{D} = F_3$$

$$(12) \qquad \frac{D'''}{D} = F_3 + F_0 F_2$$

$$(13) \qquad \frac{H'''}{D} = F_4 + 2\frac{H''}{D}F_2 + \sqrt{I_0}F_0 - 2\sqrt{I_0}F_2^2$$

to express nine of the first order scalar invariants in terms of the Cartan invariants. Here we give only five; all three of the case $[1, 1]$ and two of the case $[1, 1, 1, 1]$ in

the categorization of [15]:

$$(14) \qquad R^{;a} R_{;a} = \left(\sqrt{I_0} F_1 - 2 F_3 \right)^2$$

$$(15) \qquad \begin{aligned} R^{bc;d} R_{bc;d} = &- I_0{}^3 + 2 F_3{}^2 - 1/2 F_4{}^2 - F_0{}^2 I_0 + 2 F_0 I_0{}^2 + F_0 \sqrt{I_0} F_4 \\ &- F_2 I_0{}^{3/2} F_1 - 5 \sqrt{I_0} F_1 F_3 + F_0 F_2 \sqrt{I_0} F_1 - I_0{}^{3/2} F_4 \\ &+ \frac{23}{4} I_0 F_1{}^2 - 1/2 F_2{}^2 F_1{}^2 \end{aligned}$$

$$(16) \qquad \begin{aligned} R^{bc;d} R_{bd;c} = &1/2 I_0{}^{3/2} F_2 F_1 - 1/2 F_0 F_2 \sqrt{I_0} F_1 + \frac{9}{8} I_0 F_1{}^2 + F_3{}^2 \\ &+ 1/2 I_0{}^3 + 1/2 F_0{}^2 I_0 - F_0 I_0{}^2 - 1/2 F_0 \sqrt{I_0} F_4 \\ &- 3/2 \sqrt{I_0} F_1 F_3 + 1/2 F_2 F_4 F_1 + 1/2 I_0{}^{3/2} F_4 \end{aligned}$$

$$(17) \quad R^{;c} R_c{}^{e;f} R_{;e} R_{;f} = \left(\sqrt{I_0} F_1 - 2 F_3 \right)^3 \left(\sqrt{I_0} F_1 - F_3 \right)$$

$$(18) \quad R^a R_a{}^{b;c} R_c{}^d{}_{;b} R_{;d} = \frac{1}{8} \left(\sqrt{I_0} F_1 - 2 F_3 \right)^2 \left(9 I_0 F_1{}^2 - 16 \sqrt{I_0} F_1 F_3 + 8 F_3{}^2 \right)$$

The other scalar invariants give more expressions of the same form: they are polynomials in $\left\{ \sqrt{I_0}, F_0, F_1; F_2, F_3, F_4 \right\}$.

We have found that a representative sample of the polynomial scalar curvature invariants can be expressed as polynomials in (the square root of) the Cartan invariants. We have shown that all of the zeroth order Cartan invariants are required in this expression of the zeroth order scalar invariants, and at most the first order Cartan invariants are required for the first order scalar curvature invariants. This has been accomplished without fixing the invariant coframe.

As any combination of invariants is an invariant, this suggests that an appropriate choice of six scalar curvature invariants may be sufficient to classify the P-type I Gödel metrics, as long as one is concerned with functional dependence. For example, choosing $R = R^a{}_a$ as the functionally independent invariant, along with $\{R^{ab} R_{ab}, \ R^{ab} R_a{}^c R_{bc}\}$, and $\{R^{;a} R_{;a}, R^{bc;d} R_{bc;d}, R^{bc;d} R_{bd;c}\}$ as zeroth order and first order classifying functions, since this involves each Cartan invariant at each order at least once.

We believe this approach will be helpful for identifying the largest basis of algebraically independent scalar curvature invariants at any order. While the Newman-Penrose formalism is already employed in four dimensions for the zeroth order invariants [16], the Cartan-Karlhede algorithm enables one to reduce the number of non-zero independent components of the covariant derivatives of the curvature tensor. This can potentially simplify the form of the higher order scalar curvature invariants.

References

[1] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, and E. Herlt, 'Exact Solutions of Einstein's Field Equations 2nd Edition' Cambridge University Press (2003).

[2] F.C. Sousa, J.B. Fonseca, and C. Romero, 'Equivalence of three dimensional spacetimes', Class Quant Grav **25** 035007 (2008).

[3] Olver P *Equivalence, Invariants, and Symmetry* (Cambridge, Cambridge University Press, 1995)

[4] Spivak M *A Comprehensive Introduction to Differential Geometry, vol. 4* (Berkeley, Publish or Perish Inc, 1999)

[5] Spivak M *A Comprehensive Introduction to Differential Geometry, vol. 5* (Berkeley, Publish or Perish Inc, 1999)

[6] Ehlers J *Christoffel's Work on the Equivalence Problem for Riemannian Spaces and its Importance for Modern Field Theory of Physics* (In E.B Christoffel: The Influence of His Work on Mathematics and the Physical Sciences, Edited by P. L. Butzer and R. F. Feher) (Basel, Birkhäuser-Verlag, 1981)

[7] Cartan E *Leçons sur la Geómétrie des Espaces de Riemann* (Paris, Gauthier-Villars, 1951). English translation by J. Glazebrook, Math. Sci. Press, Brookline (1983)

[8] Karlhede A 1980 *Gen. Rel. Grav.* **12**, 693

[9] Karlhede A 2006 *Gen. Rel. Grav.* **38**, 1109

[10] A.N. Aliev, and Y. Nutku, 'Spinor formulation of topologically massive gravity', Class. Quant. Grav. **12** (1995). 2913.

[11] Penrose R and Rindler W *Spinors and Spacetime, vol. 1* (Cambridge, Cambridge University Press, 1984)

[12] Penrose R and Rindler W *Spinors and Spacetime, vol. 2* (Cambridge, Cambridge University Press, 1984)

[13] G.S. Hall, T. Morgan, and Z. Perjes 'Three-Dimensional Space-Times', Gen. Rel. Grav. **19** 1137 (1987).

[14] R. Milson, L. Wylleman, 'Three-Dimensional Spacetimes of Maximal Order', Class. Quant. Grav. **30**, 095004 (2013).

[15] A. A. Coley, A. MacDougall and D. D. McNutt, 'Basis for Scalar Curvature Invariants in Three Dimensions', CQG, **31**, 23, 235010 (2014).

[16] J. Carminati, R.G. McLenaghan, 'Algebraic invariants of the Riemann tensor in a four-dimensional Lorentzian space', J. Math. Phys. **32** pp 3135 (1991).

[17] G.E. Sneddon,